

# Feedback Capacity of Gaussian Channels Revisited

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**Abstract**—In this paper, we revisit the problem of finding the capacity of the Gaussian feedback channel where we show new results and give new proofs to existing results. In particular, we show that the channel capacity at stationarity can be found by solving a semi-definite program, and hence computationally tractable. We also give new proofs and structural results of the non stationary solution which bridges the gap between results for the stationary and non stationary feedback channel capacity.

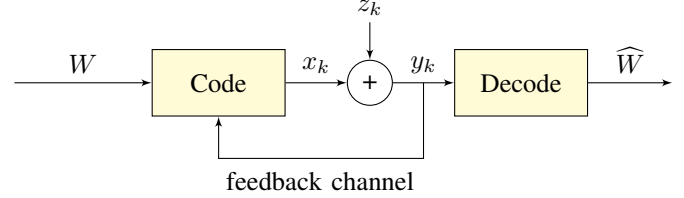


Figure 1. The system studied in the paper. The feedback channel is assumed to be noiseless and the measurement noise  $z_k$  is given by some colored Gaussian noise with statistics known at the transmitter and receiver.

## I. INTRODUCTION

### A. Background and Previous Work

We revisit the problem of communication over a Gaussian feedback channel with colored noise given by single-input-single-output (SISO) dynamical system of a finite order (see Figure 1). The noise process is given by  $z = \mathbf{H}u$ . Let  $\mathbf{H}$  be a linear dynamical system given by the state space equations

$$\begin{aligned} s_{k+1} &= F s_k + G u_k \\ z_k &= H s_k + u_k \\ s_1 &= 0 \\ u_k &\sim \mathcal{N}(0, 1) \\ \mathbf{E}(u_k s_1) &= 0 \quad \forall k \\ \mathbf{E}(u_k u_l^T) &= 0 \quad \forall k, l \mid k \neq l \end{aligned}$$

where  $u_k$ ,  $s_k$ , and  $z_k$  take values in  $\mathbb{R}$ ,  $\mathbb{R}^m$ , and  $\mathbb{R}$ , respectively. Let  $W$  be the message to be transmitted and consider the Gaussian communication channel

$$y_k = x_k + z_k$$

over a time horizon  $k = 1, \dots, n$ , where  $x_k = f_k(W, x^{k-1}, y^{k-1})$  is the transmitted signal over the channel with the average power constraint

$$\frac{1}{n} \sum_{k=1}^n \mathbf{E}(|x_k|^2) \leq P,$$

$y$  is the measurement signal at the receiver. The average channel capacity

$$C_n = \sup_f \frac{1}{n} \mathbf{I}(W; y^n).$$

In particular, find  $C = \lim_{n \rightarrow \infty} C_n$ .

It's well known that for the case where the noise is white (that is  $\mathbf{H}$  is the identity operator), feedback does not improve on the capacity of the channel. However, feedback does indeed increase the capacity when the noise is colored. In the seminal work by Cover [1], it was shown that the optimal strategy to

maximize the  $C_n$  is given by  $x^n = B_n z^n + v^n$ , where  $B_n$  is strictly lower triangular and  $v^n$  is Gaussian white noise. The capacity  $C_n$  was shown to be possible to calculate using semi-definite programming [2]. However, the semi-definite program size grows linearly with  $n$ , and it's not possible to use the approach of [1] to find the stationary channel capacity as  $n$  goes to infinity. In [3], it was shown that the channel capacity at stationarity is given by

$$C = \frac{1}{2} \log_2(Y)$$

and  $Y$  can be obtained by solving the nonconvex optimization problem

$$\begin{aligned} \max_{X, \Sigma \succeq 0} \quad & Y \\ \text{s. t.} \quad & P \geq X \Sigma X^T \\ & \Sigma = F \Sigma F^T + G G^T - \Gamma Y \Gamma^T \\ & \Gamma = (F \Sigma (X + H)^T + G) Y^{-1} \\ & Y = (X + H) \Sigma (X + H)^T + 1 \end{aligned}$$

The solution relies on considering the stationary problem directly instead of solving the problem over a finite horizon  $n$  and then letting  $n \rightarrow \infty$ . The stationarity property in turn allows for using problem formulations in the frequency domain with some revealing structure that are not obvious to see in the time domain. However, a solution to the above optimization problem is intractable in practice and one needs another approach in order to get a practical solution.

### B. Contributions

This paper revisits the problem of feedback capacity of the Gaussian channel where we show new results and give new proofs to existing results. In particular, we show that the channel capacity is given by

$$C = \frac{1}{2} \log_2(Y)$$

where  $Y$  is the value of the semi-definite program

$$\begin{aligned} & \sup_{K, \Sigma \succeq 0} Y \\ & \text{s. t.} \\ & 0 \preceq \begin{pmatrix} P & K \\ K^\top & \Sigma \end{pmatrix} \\ & 0 \preceq \begin{pmatrix} F\Sigma F^\top - \Sigma + GG^\top & FK^\top + F\Sigma H^\top + G \\ (FK^\top + F\Sigma H^\top + G)^\top & Y \end{pmatrix} \\ & Y = KH^\top + HK^\top + H\Sigma H^\top + P + 1 \end{aligned}$$

We also unify the solutions of [1] and [3] by giving a new and simpler proof completely based on a state space approach in the time domain, without resorting to the frequency domain. The state space domain approach provides a new proof of the optimality of the linear feedback strategy in maximizing the capacity  $C_n$  over a finite time horizon  $n$ , rendering results of [1]. The solution of the stationary feedback capacity  $C = \lim_{n \rightarrow \infty} C_n$  follows from the finite horizon case, which includes the solution found in [3] and extends this computationally intractable solution to a tractable one based on convex optimization.

## II. PRELIMINARIES

### A. Notation

$\mathbb{N}$	The set of positive integers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{C}$	The set of complex numbers.
$\mathbb{S}^n$	The set of $n \times n$ symmetric matrices.
$\mathbb{S}_+^n$	The set of $n \times n$ symmetric positive semidefinite matrices.
$\mathbb{S}_{++}^n$	The set of $n \times n$ symmetric positive definite matrices.
$\succeq$	$A \succeq B \iff A - B \in \mathbb{S}_+^n$ .
$\succ$	$A \succ B \iff A - B \in \mathbb{S}_{++}^n$ .
$s^k$	$s^k = (s_1, s_2, \dots, s_k)$ and $s^0 \triangleq 0$ .
$\ s\ $	For a sequence $s = (s_1, s_2, \dots, s_k)$ , $\ s\ ^2 = \sum_{i=1}^k  s_i ^2$ .

### B. System Theory

The material here can be found in [4].

**Definition 1** (Detectability). Let  $F \in \mathbb{R}^{m \times m}$  and  $H \in \mathbb{R}^{p \times m}$ . The pair of matrices  $(H, F)$  is detectable if

$$\begin{pmatrix} F - \lambda I \\ H \end{pmatrix}$$

has full column rank for all  $\lambda \in \mathbb{C}$  such that  $|\lambda| \geq 1$ .

**Definition 2** (Observability). Let  $F \in \mathbb{R}^{m \times m}$  and  $H \in \mathbb{R}^{p \times m}$ . The pair of matrices  $(H, F)$  is observable if

$$\begin{pmatrix} F - \lambda I \\ H \end{pmatrix}$$

has full column rank for all  $\lambda \in \mathbb{C}$ .

**Definition 3** (Controllability). Let  $F \in \mathbb{R}^{m \times m}$  and  $G \in \mathbb{R}^{m \times r}$ . The pair of matrices  $(F, G)$  is controllable if

$$(F - \lambda I \quad G)$$

has full row rank for all  $\lambda \in \mathbb{C}$ .

**Definition 4** (Stability). Let  $F \in \mathbb{R}^{m \times m}$ . The matrix  $F$  is stable if and only if its eigenvalues have modulus strictly less than 1.

**Proposition 1.** Let  $Q \succeq 0$  and  $F$  stable. Then, the unique positive semi-definite solution  $\Sigma$  to the Riccati equation

$$\Sigma = F\Sigma F^\top + Q$$

is invertible if  $(F, Q)$  is controllable.

*Proof:* Consult [4]. ■

**Proposition 2.** Let  $F$  be a stable matrix. Then, the pair  $(F, G)$  is controllable if and only if there does not exist a vector  $x$  and a scalar  $\lambda \in \mathbb{C}$  such that  $x^\top F = \lambda x^\top$  and  $x^\top G = 0$ .

*Proof:* Consult [4]. ■

### C. Optimal Estimation of Gaussian Processes

Consider a linear dynamical system  $\mathbf{H}$  given by the state space equations

$$\begin{aligned} s_{k+1} &= Fs_k + Gu_k \\ z_k &= Hs_k + Eu_k \\ s_1 &\sim \mathcal{N}(0, S_1) \\ u_k &\sim \mathcal{N}(0, 1) \\ \mathbf{E}(u_k s_1^\top) &= 0 \quad \forall k \\ \mathbf{E}(u_k u_l^\top) &= 0 \quad \forall k, l \mid k \neq l \end{aligned} \tag{1}$$

where  $u_k$ ,  $s_k$ , and  $z_k$  take values in  $\mathbb{R}$ ,  $\mathbb{R}^m$ , and  $\mathbb{R}$ , respectively. Let  $\hat{s}_k = \mu_k(z^{k-1})$  be an estimate of  $s_k$  based on the measurements  $z^{k-1}$  and  $\tilde{s}_k = s_k - \hat{s}_k$ . Suppose that we want to minimize the average estimation error

$$\frac{1}{n} \sum_{k=1}^n \mathbf{E}(|\tilde{s}_k|^2)$$

It's well known that the optimal estimator is given by  $\hat{s}_k = \mathbf{E}(s_k | z^{k-1})$  which obeys the the optimal Kalman filter recursions [5]

$$\begin{aligned} S_k &= \mathbf{E}(\tilde{s}_k \tilde{s}_k^\top) \\ S_1 &= \mathbf{E}(s_1 s_1^\top) \\ S_{k+1} &= FS_k F^\top + GG^\top \\ &\quad - (FS_k H^\top + GE^\top)(HS_k H^\top + EE^\top)^{-1} \\ &\quad \times (HS_k F^\top + EG^\top) \\ K_k &= (FS_k H^\top + GE^\top)(HS_k H^\top + EE^\top)^{-1} \\ \hat{s}_{k+1} &= F\hat{s}_k + K_k(z_k - H\hat{s}_k) \\ \tilde{s}_{k+1} &= (F - K_k H)\tilde{s}_k + Gu_k - K_k Eu_k \\ \tilde{z}_k &= H\tilde{s}_k + Eu_k \end{aligned} \tag{2}$$

where we have assumed that  $(HS_k H^\top + EE^\top)$  is invertible for all  $k \in \mathbb{N}$  (note that this assumption is not restrictive as we may replace the inverse with the Moore-Penrose pseudo inverse). A property of the Kalman filter is that the innovations  $\tilde{z}_k = z_k - H\hat{s}_k = H\tilde{s}_k + Eu_k$  are independent for all  $k \in \mathbb{N}$ .

#### D. Entropy Properties of Gaussian Variables and Processes

The entropy of the Gaussian process given by (1) over a time horizon  $k = 1, \dots, n$  is  $h(z^n)$  which may be rewritten as a sum of conditional entropies using the entropy chain rule

$$h(z^n) = \sum_{k=1}^n h(z_k | z^{k-1})$$

with  $z^k \triangleq 0$  if  $k = 0$ . The entropy rate is given by

$$h(z) = \lim_{n \rightarrow \infty} \frac{1}{n} h(z^n)$$

**Proposition 3.** Consider a linear dynamical system  $\mathbf{H}$  given by (1) over a finite time horizon  $k = 1, \dots, n$ . The entropy of  $\mathbf{H}$  is given by

$$h(z^n) = \frac{1}{2} \sum_{k=1}^n \log_2 (2\pi e (HS_k H^\top + EE^\top))$$

where  $S_k$  is given by the recursion

$$\begin{aligned} S_1 &= \mathbf{E}(s_1 s_1^\top) \\ S_{k+1} &= FS_k F^\top + GG^\top \\ &\quad - (FS_k H^\top + G)(HS_k H^\top + EE^\top)^{-1} \\ &\quad \times (HS_k F^\top + G^\top) \end{aligned}$$

Furthermore, if  $(H, F)$  is detectable, then the stationary entropy rate is given by

$$h(z) = \lim_{n \rightarrow \infty} \frac{1}{n} h(z^n) = \frac{1}{2} \log_2 (2\pi e (HSH^\top + EE^\top))$$

where  $S$  is the unique solution to the Riccati equation

$$\begin{aligned} S &= FSF^\top + GG^\top \\ &\quad - (FSH^\top + G)(HSH^\top + EE^\top)^{-1} \\ &\quad \times (HSF^\top + G^\top) \end{aligned} \quad (3)$$

*Proof:* See the appendix. ■

### III. FEEDBACK CAPACITY OF GAUSSIAN CHANNELS WITH COLORED NOISE

Let  $\mathbf{H}$  be a linear dynamical system given by the state space equations

$$\begin{aligned} s_{k+1} &= Fs_k + Gu_k \\ z_k &= Hs_k + u_k \\ s_1 &= 0 \\ u_k &\sim \mathcal{N}(0, 1) \\ \mathbf{E}(u_k s_1) &= 0 \quad \forall k \\ \mathbf{E}(u_k u_l^\top) &= 0 \quad \forall k, l \mid k \neq l \end{aligned} \quad (4)$$

where  $u_k$ ,  $s_k$ , and  $z_k$  take values in  $\mathbb{R}$ ,  $\mathbb{R}^m$ , and  $\mathbb{R}$ , respectively.

We consider the following problem.

**Problem 1.** Let  $W$  be the message to be transmitted and consider the Gaussian communication channel

$$y_k = x_k + z_k$$

over a time horizon  $k = 1, \dots, n$ , where  $x_k = f_k(W, x^{k-1}, y^{k-1})$  is the transmitted signal over the channel with the average power constraint

$$\frac{1}{n} \sum_{k=1}^n \mathbf{E}(|x_k|^2) \leq P,$$

$y$  is the measurement signal at the receiver, and  $z$  is the Gaussian measurement noise process described by a filter  $\mathbf{H}$  given by Equation (4), where the pair  $(F, G)$  is controllable. Find the average channel capacity

$$C_n = \sup_f \frac{1}{n} \mathbf{I}(W; y^n).$$

In particular, find  $C = \lim_{n \rightarrow \infty} C_n$ .

It has been shown in [1] that for a noise sequence  $z^n$ , the mutual information  $\mathbf{I}(W; y^n)$  is given by

$$\mathbf{I}(W; y^n) = h(y^n) - h(z^n)$$

To make this paper self contained, we state this result and give the proof in the appendix.

**Proposition 4.** Consider the Gaussian feedback channel as described in Problem 1. Then,

$$\mathbf{I}(W; y^n) = h(y^n) - h(z^n)$$

*Proof:* See the appendix. ■

It was shown in [1] that the optimal input sequence  $x^n$  has the form  $x^n = B_n z^n + v^n$  where  $B_n \in \mathbb{R}^{n \times n}$  is strictly lower triangular,  $v_n$  is a Gaussian sequence with a covariance  $V_n \in \mathbb{S}_+^n$ , and the pair  $(B_n, V_n)$  satisfies the power constraint

$$\text{Tr}(B_n Z_n B_n^\top + V_n) = \mathbf{E}(|x^k|^2) \leq nP$$

Now the mutual information between  $v^n$  and  $y^n$  is given by

$$\begin{aligned} \mathbf{I}(v^n; y^n) &= \frac{1}{2} \log_2 \frac{\det(V_n + (I + B_n)Z_n(I + B_n)^\top)}{\det(Z_n)} \\ &= \frac{1}{2} \log_2 \det(V_n + (I + B_n)Z_n(I + B_n)^\top) \\ &\quad - \frac{1}{2} \log_2 \det(Z_n) \\ &=: \mathcal{I}(B_n, V_n, Z_n) \end{aligned}$$

For a noise sequence  $z^n$  with covariance  $Z_n$ , the average feedback capacity over the time horizon  $1, \dots, n$  is given by

$$C_n = \max_{\substack{B_n, V_n \\ \text{Tr}(B_n Z_n B_n^\top + V_n) \leq nP}} \frac{1}{n} \mathcal{I}(B_n, V_n, Z_n)$$

where the maximum is taken over  $B_n \in \mathbb{R}^{n \times n}$  being strictly lower triangular.

In [3], by using the results of [1] above, it was shown that the optimal affine strategy of the transmitted symbols  $x_k$  to maximize the capacity at stationarity (that is, as  $n \rightarrow \infty$ ), is given by

$$x_k = X(s_k - \mathbf{E}(s_k | y^{k-1})) + v_k$$

where  $v$  is a white Gaussian process independent of  $u$  and with  $v_k \sim \mathcal{N}(0, V)$ , and  $X^\top$  is some vector in  $\mathbb{R}^m$ . In fact,

it was shown that  $v = 0$  is optimal, but we will keep it as an optimization parameter as it will simplify the optimization problem considerably.

We give here new (and shorter) proofs that summarize the results of [1] and [3]. It also generalizes the results of [3] to the (non-stationary) finite-horizon case.

**Theorem 1.** *The optimal average feedback capacity in Problem 1 over a finite horizon  $k = 1, \dots, n$  is achieved by  $x_k = X_k \tilde{s}_k + v_k$  for some set of vectors  $\{X_k^\top\}$ , where  $\hat{s}_k = \mathbf{E}(s_k | y^{k-1})$ ,  $\tilde{s}_k = s_k - \hat{s}_k$ , and  $\{v_k\}$  is an uncorrelated Gaussian sequence. The capacity is given by*

$$C_n = \sup_{\substack{X_1, \dots, X_n \\ V_1, \dots, V_n \geq 0}} \frac{1}{2n} \sum_{k=1}^n \log_2(Y_k)$$

subject to  $\Sigma_1 = 0$ ,

$$\begin{aligned} P &\geq X_k \Sigma_k X_k^\top + V_k, \\ Y_k &= (X_k + H) \Sigma_k (X_k + H)^\top + V_k + 1, \end{aligned} \quad (5)$$

$$\Gamma_k = (F \Sigma_k (X_k + H)^\top + G) Y_k^{-1}$$

and

$$\begin{aligned} \Sigma_{k+1} &= (F - \Gamma_k (X_k + H)) \Sigma_k (F - \Gamma_k (X_k + H))^\top \\ &\quad + (G - \Gamma_k) (G - \Gamma_k)^\top + \Gamma_k V_k \Gamma_k^\top \\ &= F \Sigma_k F^\top + G G^\top - \Gamma_k Y_k \Gamma_k^\top \end{aligned} \quad (6)$$

*Proof:* See the appendix.

At stationarity, the variance of  $\tilde{y}_k$  is given by

$$Y = (X + H) \Sigma (X + H)^\top + V + 1 \quad (7)$$

where

$$\begin{aligned} X &= \lim_{k \rightarrow \infty} X_k, \\ V &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n V_k, \end{aligned}$$

$\Sigma$  is the unique solution to the Riccati equation

$$\begin{aligned} \Sigma &= (F - \Gamma(X + H)) \Sigma (F - \Gamma(X + H))^\top \\ &\quad + (G - \Gamma) (G - \Gamma)^\top + \Gamma V \Gamma^\top \\ &= F \Sigma F^\top + G G^\top - \Gamma Y \Gamma^\top \end{aligned} \quad (8)$$

and

$$\Gamma = (F \Sigma (X + H)^\top + G) Y^{-1}$$

Recall that the power constraint at stationarity is given by

$$P \geq \mathbf{E}(x_k^2) = X \Sigma X^\top + V$$

The average feedback channel capacity is then given by

$$\lim_{n \rightarrow \infty} C_n = \frac{1}{2} \log_2(Y)$$

and we arrive at the following optimization problem for finding the maximum capacity at stationarity:

$$\begin{aligned} \max_{X, \Sigma \geq 0, V \geq 0} \quad & \frac{1}{2} \log_2(Y) \\ \text{s. t.} \quad & P \geq X \Sigma X^\top + V \\ & \Sigma = F \Sigma F^\top + G G^\top - \Gamma Y \Gamma^\top \\ & \Gamma = (F \Sigma (X + H)^\top + G) Y^{-1} \\ & Y = (X + H) \Sigma (X + H)^\top + V + 1 \end{aligned}$$

Note also that taking  $V = 0$  renders the same (nonconvex) optimization problem as that in [3].

Since  $Y$  is a scalar, we may maximize  $Y$  instead of its logarithm. We can also restrict  $V$  to be strictly positive and take the infimum over  $V$  instead of the maximum. This is useful as we will show later. The equivalent optimization problem becomes

$$\begin{aligned} \sup_{V > 0} \sup_{X, \Sigma \geq 0} \quad & Y \\ \text{s. t.} \quad & P \geq X \Sigma X^\top + V \\ & \Sigma = F \Sigma F^\top + G G^\top - \Gamma Y \Gamma^\top \\ & \Gamma = (F \Sigma (X + H)^\top + G) Y^{-1} \\ & Y = (X + H) \Sigma (X + H)^\top + V + 1 \end{aligned} \quad (9)$$

Now we turn to the Riccati equation (8) and utilize that  $V > 0$  to show that the pair

$$(F - \Gamma(X + H), (G - \Gamma)(G - \Gamma)^\top + \Gamma V \Gamma^\top)$$

is controllable.

**Lemma 1.** *Suppose that  $(F, G)$  is controllable and  $V > 0$ . Then, the pair*

$$(F - \Gamma(X + H), (G - \Gamma)(G - \Gamma)^\top + \Gamma V \Gamma^\top)$$

*is controllable.*

*Proof:* See the appendix. ■

Now we can use Lemma 1 to prove the following.

**Lemma 2.** *Suppose that  $(F, G)$  is controllable and  $V > 0$  and consider the Riccati equation*

$$\begin{aligned} \Sigma &= (F - \Gamma(X + H)) \Sigma (F - \Gamma(X + H))^\top \\ &\quad + (G - \Gamma) (G - \Gamma)^\top + \Gamma V \Gamma^\top \\ &= F \Sigma F^\top + G G^\top - \Gamma Y \Gamma^\top \end{aligned}$$

*Then,  $\Sigma \succ 0$ .*

*Proof:* Lemma 1 together with Proposition 1,  $\Sigma$  must be invertible and thus strictly positive definite. ■

Since Lemma 2 established the invertibility of  $\Sigma$ , we see that optimization problem (9) is equivalent to that of optimizing over strictly positive definite matrices  $\Sigma$ :

$$\begin{aligned} \sup_{V > 0} \sup_{X, \Sigma \succ 0} \quad & Y \\ \text{s. t.} \quad & P \geq X \Sigma X^\top + V \\ & \Sigma = F \Sigma F^\top + G G^\top - \Gamma Y \Gamma^\top \\ & \Gamma = (F \Sigma (X + H)^\top + G) Y^{-1} \\ & Y = (X + H) \Sigma (X + H)^\top + V + 1 \end{aligned} \quad (10)$$

Optimization problem (10) can now be transformed to a semi-definite program. The trick is to first eliminate the dependence on  $V$  and obtain desired inequalities instead of equalities. Then, by making a variable substitution according to  $K = X \Sigma$ , which is possible since  $\Sigma$  is invertible, we will be able to use a Schur complement argument in order to transform the constraints in (10) into a set of linear matrix inequalities (LMI:s).

We are now ready to state the main result of this paper.

**Theorem 2.** *The feedback capacity of the Gaussian channel is given by*

$$C = \frac{1}{2} \log_2(Y)$$

where  $Y$  the optimal value of

$$\begin{aligned} & \sup_{K, \Sigma \succ 0} Y \\ & \text{s. t.} \\ & 0 \preceq \begin{pmatrix} P & K \\ K^\top & \Sigma \end{pmatrix} \\ & 0 \preceq \begin{pmatrix} F\Sigma F^\top - \Sigma + GG^\top & FK^\top + F\Sigma H^\top + G \\ (FK^\top + F\Sigma H^\top + G)^\top & Y \end{pmatrix} \\ & Y = KH^\top + HK^\top + H\Sigma H^\top + P + 1 \end{aligned} \quad (11)$$

*Proof:* See the appendix. ■

#### IV. NUMERICAL EXAMPLE

Consider a communication feedback channel with Gaussian noise given by a first order process according to

$$\begin{aligned} s_{k+1} &= -\beta s_k + u_k \\ z_k &= (\alpha - \beta)s_k + u_k \\ y_k &= x_k + z_k \end{aligned}$$

with  $\alpha \in [-1, 1]$ ,  $\beta \in (-1, 1)$ ,  $u_k \sim \mathcal{N}(0, 1)$ , and

$$\sigma = \text{sign}(\beta - \alpha) = \begin{cases} 1 & \text{if } \beta > \alpha \\ 0 & \text{if } \beta = \alpha \\ -1 & \text{if } \beta < \alpha \end{cases}$$

In [3], it was shown that the feedback capacity of the above channel with a power constraint  $\mathbf{E}(x_k^2) \leq P$  is given by

$$-\log_2(r)$$

where  $r$  is the unique positive real root of the polynomial equation

$$(\beta^2 P + \alpha^2)r^4 + 2\sigma(\alpha + \beta P)r^3 + (P + 1 - \alpha^2)r^2 - 2\sigma\alpha r - 1 = 0$$

This can be compared to the solution of the semi-definite optimization problem (11). By noting that  $F = -\beta$ ,  $G = 1$ , and  $H = \alpha - \beta$ , one can verify numerically that the channel capacity  $\frac{1}{2} \log_2(Y)$  coincides with that of the polynomial solution  $-\log_2(r)$  (see the Matlab code in the appendix, where we used CVX, a package for specifying and solving convex programs [6], [7]).

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#### APPENDIX

##### Proof of Proposition 3

Note that  $\hat{s}_k = \mathbf{E}(s_k | z^{k-1})$  and  $\tilde{s}_k = s_k - \hat{s}_k$  are given by the Kalman filter (2), so  $z_k - \mathbf{E}(z_k | z^{k-1}) = \tilde{z}_k$  is the Gaussian output estimation error given by the Kalman filter (2). Thus, the entropy chain rule gives the equality

$$h(z^n) = \sum_{k=1}^n h(\tilde{z}_k) = \frac{1}{2} \sum_{k=1}^n \log_2(2\pi e(HS_k H^\top + EE^\top))$$

The stationary entropy rate is then given by

$$h(z) = \lim_{n \rightarrow \infty} \frac{1}{n} h(z^n) = \frac{1}{2} \log_2(2\pi e(HSH^\top + EE^\top))$$

where  $S$  the unique solution to the Riccati equation (3).

##### Proof of Proposition 4

$$I(W; y^n) = h(y^n) - h(y^n | W)$$

$$\begin{aligned} &= h(y^n) - \sum_{k=1}^n h(y_k | W, y^{k-1}) \\ &= h(y^n) - \sum_{k=1}^n h(y_k | W, y^{k-1}, x_k(W, y^{k-1}), x^{k-1}) \end{aligned} \quad (12)$$

$$= h(y^n) - \sum_{k=1}^n h(y_k | W, z^{k-1}, x_k, x^{k-1}) \quad (13)$$

$$= h(y^n) - \sum_{k=1}^n h(z_k | W, z^{k-1}, x_k, x^{k-1}) \quad (14)$$

$$= h(y^n) - \sum_{k=1}^n h(z_k | W, z^{k-1}) \quad (15)$$

$$\begin{aligned} &= h(y^n) - h(z^n | W) \\ &= h(y^n) - h(z^n) \end{aligned} \quad (16)$$

where (12) follows from the fact that  $x_i = f_i(W, x^{i-1}, y^{i-1})$  and so  $x_i$  is determined by  $(W, y^{i-1})$ , (13) follows from the equality  $y^{k-1} = x^{k-1} + z^{k-1}$ , (14) follows from  $y_k = x_k + z_k$ , (15) follows from the fact that  $x^k$  is determined from  $W$  and  $z^{k-1}$  by the recursion  $x_i = f_i(W, x^{i-1}, x^{i-1} + z^{i-1})$ , and (16) follows from the independence between  $z^n$  and  $W$ .

##### Proof of Theorem 1

According to Proposition 4, we have that

$$I(W; y^n) = h(y^n) - h(z^n).$$

Let  $\hat{z}_k = \mathbf{E}(z_k | z^{k-1})$  and  $\tilde{z}_k = z_k - \hat{z}_k$ . We have that

$$h(z^n) = \sum_{k=1}^n h(z_k | z^{k-1}) = \sum_{k=1}^n h(\tilde{z}_k).$$

Since  $s_1 = 0$ , we get  $\tilde{z}_1 = z_1 = u_1$ . We also have that  $\mathbf{E}(z_2 | z_1) = Hs_2$ , and so  $\tilde{z}_2 = u_2$ . Inductively, we get that  $\tilde{z}_k = u_k$  for  $k = 1, \dots, n$ . Thus,

$$h(z^n) = \sum_{k=1}^n h(\tilde{z}_k) = \sum_{k=1}^n h(u_k) = \frac{1}{2} \sum_{k=1}^n \log_2(2\pi e)$$

Let  $\hat{s}_k = \mathbf{E}(s_k | y^{k-1})$ ,  $\tilde{s}_k = s_k - \hat{s}_k$ , and

$$\tilde{y}_k = y_k - H\hat{s}_k = x_k + H\tilde{s}_k + u_k.$$

Note that

$$h(y^n) = \sum_{k=1}^n h(y_k | y^{k-1}) \quad (17)$$

$$= \sum_{k=1}^n h(\tilde{y}_k | y^{k-1}) \quad (18)$$

$$\leq \sum_{k=1}^n h(\tilde{y}_k) \quad (19)$$

$$\leq \frac{1}{2} \sum_{k=1}^n \log_2(2\pi e \mathbf{E}(\tilde{y}_k^2)) \quad (20)$$

where (19) follows from the fact that conditioning only reduces entropy (with equality if  $\tilde{y}_k$  is independent of  $y^{k-1}$ ) and (20) follows from the fact that for a fixed covariance, the Gaussian distribution has the maximum entropy. We will now show that

$$Y_k \triangleq \mathbf{E}(\tilde{y}_k^2)$$

can be achieved by the transmission strategy  $x_k = X_k \tilde{s}_k + v_k$ . First note that  $\mathbf{E}(\tilde{y}_k^2) = \mathbf{E}((x_k + H\tilde{s}_k + u_k)^2) = \mathbf{E}((x_k + H\tilde{s}_k)^2) + \mathbf{E}(u_k^2)$  since  $u_k$  is independent of  $x_k$  and  $\tilde{s}_k$ . Now let

$$\mathbf{E}\left(\begin{pmatrix} \tilde{s}_k \\ x_k \end{pmatrix} \begin{pmatrix} \tilde{s}_k \\ x_k \end{pmatrix}^\top\right) = \begin{pmatrix} \Sigma_k & \Psi_k \\ \Psi_k^\top & \Xi_k \end{pmatrix} \quad (21)$$

be an achievable covariance matrix by some strategy  $x_k$ . Then,

$$\begin{aligned} Y_k &= \mathbf{E}(\tilde{y}_k^2) \\ &= \mathbf{E}((x_k + H\tilde{s}_k)^2) + \mathbf{E}(u_k^2) \\ &= \mathbf{E}\left(\begin{pmatrix} \tilde{s}_k \\ x_k \end{pmatrix}^\top \begin{pmatrix} H^\top H & H^\top \\ H & 1 \end{pmatrix} \begin{pmatrix} \tilde{s}_k \\ x_k \end{pmatrix}\right) + 1 \\ &= \mathbf{E}\left(\text{Tr}\left(\begin{pmatrix} H^\top H & H^\top \\ H & 1 \end{pmatrix} \begin{pmatrix} \tilde{s}_k \\ x_k \end{pmatrix} \begin{pmatrix} \tilde{s}_k \\ x_k \end{pmatrix}^\top\right)\right) + 1 \\ &= \text{Tr}\left(\begin{pmatrix} H^\top H & H^\top \\ H & 1 \end{pmatrix} \begin{pmatrix} \Sigma_k & \Psi_k \\ \Psi_k^\top & \Xi_k \end{pmatrix}\right) + 1 \end{aligned}$$

The Schur complement in  $\Xi_k$  of

$$\begin{pmatrix} \Sigma_k & \Psi_k \\ \Psi_k^\top & \Xi_k \end{pmatrix} \succeq 0,$$

is given by

$$\Phi_k \triangleq \Xi_k - \Psi_k^\top \Sigma_k^{-1} \Psi_k \succeq 0.$$

By taking  $x_k = X_k \tilde{s}_k + v_k$  with  $X_k = \Psi_k^\top \Sigma_k^{-1}$  and  $v_k \sim \mathcal{N}(0, \Phi_k)$  independent of  $\tilde{s}_k$ ,  $u_k$ , and  $v_l$  for  $l \neq k$ , we will get a sequence of pairs  $(\tilde{s}_k, x_k)$  that renders the covariance matrix (21). Also, since this strategy is linear and  $v$  is a Gaussian process,  $\tilde{y}$  is Gaussian such that  $\tilde{y}_k$  is independent of  $y^{k-1}$ ,

and the entropy upper bound (20) is achieved. The mutual information becomes

$$\begin{aligned} I(W; y^n) &= h(y^n) - h(z^n) \\ &= \frac{1}{2} \sum_{k=1}^n \log_2(2\pi e \mathbf{E}(\tilde{y}_k^2)) - \frac{1}{2} \sum_{k=1}^n \log_2(2\pi e) \\ &= \frac{1}{2} \sum_{k=1}^n \log_2(2\pi e Y_k) - \frac{1}{2} \sum_{k=1}^n \log_2(2\pi e) \\ &= \frac{1}{2} \sum_{k=1}^n \log_2(Y_k) \end{aligned}$$

and the average feedback channel capacity is

$$C_n = \sup_f \frac{1}{n} I(W; y^n) = \max_{\substack{X_1, \dots, X_n \\ V_1, \dots, V_n \geq 0}} \frac{1}{2n} \sum_{k=1}^n \log_2(Y_k)$$

Now for  $x_k = X_k \tilde{s}_k + v_k$ , we have that

$$\tilde{y}_k = y_k - H\hat{s}_k = (X_k + H)\tilde{s}_k + v_k + u_k.$$

Let

$$\begin{aligned} \Gamma_k &\triangleq \mathbf{E}((F\tilde{s}_k + Gu_k)\tilde{y}_k) (\mathbf{E}(\tilde{y}_k^2))^{-1} \\ &= (F\Sigma_k(X_k + H)^\top + G)Y_k^{-1} \end{aligned}$$

Then,

$$\mathbf{E}(F\tilde{s}_k + Gu_k | \tilde{y}_k) = \Gamma_k \tilde{y}_k$$

The dynamics of  $\hat{s}_k$  and  $\tilde{s}_k$  are given by

$$\begin{aligned} \hat{s}_{k+1} &= \mathbf{E}(Fs_k + Gu_k | y^k) \\ &= \mathbf{E}(Fs_k + Gu_k | y_k, y^{k-1}) \\ &= \mathbf{E}(F(\hat{s}_k + \tilde{s}_k) + Gu_k | \tilde{y}_k, y^{k-1}) \\ &= F\hat{s}_k + \mathbf{E}(F\tilde{s}_k + Gu_k | \tilde{y}_k) \\ &= F\hat{s}_k + \mathbf{E}(F\tilde{s}_k + Gu_k | \tilde{y}_k) \\ &= F\hat{s}_k + \Gamma_k \tilde{y}_k \end{aligned}$$

where the fourth equality above follows from the orthogonality between  $(u_k, \tilde{s}_k, \tilde{y}_k)$  and  $y^{k-1}$ . Hence, the error dynamics become

$$\begin{aligned} \tilde{s}_{k+1} &= F\tilde{s}_k - \Gamma_k \tilde{y}_k + Gu_k \\ \tilde{y}_k &= (X_k + H)\tilde{s}_k + v_k + u_k \end{aligned}$$

which implies that

$$Y_k = (X_k + H)\Sigma_k(X_k + H)^\top + V_k + 1.$$

Finally, we have that  $\Sigma_1 = \mathbf{E}(\tilde{s}_1 \tilde{s}_1^\top) = \mathbf{E}(s_1 s_1^\top) = 0$  and the

recursion of  $\Sigma_k$

$$\begin{aligned}
\Sigma_{k+1} &= (F - \Gamma_k(X_k + H))\Sigma_k(F - \Gamma_k(X_k + H))^\top \\
&\quad + (G - \Gamma_k)(G - \Gamma_k)^\top + \Gamma_k V_k \Gamma_k^\top \\
&= F\Sigma_k F^\top - \Gamma_k(X_k + H)\Sigma_k F - F^\top \Sigma_k (X_k + H)^\top \Gamma_k^\top \\
&\quad + \Gamma_k(X_k + H)\Sigma_k(X_k + H)^\top \Gamma_k^\top + GG^\top - \Gamma_k G^\top \\
&\quad - G\Gamma_k^\top + \Gamma_k \Gamma_k^\top + \Gamma_k V_k \Gamma_k^\top \\
&= F\Sigma_k F^\top - \Gamma_k(X_k + H)\Sigma_k F - F^\top \Sigma_k (X_k + H)^\top \Gamma_k^\top \\
&\quad + \Gamma_k(X_k + H)\Sigma_k(X_k + H)^\top \Gamma_k^\top + GG^\top - \Gamma_k G^\top \\
&\quad - G\Gamma_k^\top + \Gamma_k \Gamma_k^\top \\
&\quad + \Gamma_k(Y_k - (X_k + H)\Sigma_k(X_k + H)^\top - 1)\Gamma_k^\top \\
&= F\Sigma_k F^\top + GG^\top - \Gamma_k((X_k + H)\Sigma_k F + G^\top) \\
&\quad - (F^\top \Sigma_k(X_k + H)^\top + G)\Gamma_k^\top + \Gamma_k Y_k \Gamma_k^\top \\
&= F\Sigma_k F^\top + GG^\top - \Gamma_k Y_k Y_k^{-1}((X_k + H)\Sigma_k F + G^\top) \\
&\quad - (F^\top \Sigma_k(X_k + H)^\top + G)Y_k^{-1}Y_k \Gamma_k^\top + \Gamma_k Y_k \Gamma_k^\top \\
&= F\Sigma_k F^\top + GG^\top - \Gamma_k Y_k \Gamma_k^\top - \Gamma_k Y_k \Gamma_k^\top + \Gamma_k Y_k \Gamma_k^\top \\
&= F\Sigma_k F^\top + GG^\top - \Gamma_k Y_k \Gamma_k^\top
\end{aligned}$$

*Proof of Lemma 1*

Suppose that  $x$  is such that  $x^\top(G - \Gamma)(G - \Gamma)^\top = 0$  and  $x^\top \Gamma V \Gamma^\top = 0$ . Then, we must have  $x^\top \Gamma = 0$  since  $V > 0$ . Also,  $x^\top(G - \Gamma)(G - \Gamma)^\top = 0$  implies that  $x^\top(G - \Gamma) = 0$  and thus,  $x^\top G = 0$ . Now  $(F, G)$  is controllable, and Proposition 2 implies that it doesn't exist a number  $\lambda \in \mathbb{C}$  such that  $\lambda x^\top = x^\top F = x^\top(F - \Gamma(X_k + H))$ . Hence, using Proposition 2 again, we conclude that

$$(F - \Gamma(X_k + H), (G - \Gamma)(G - \Gamma)^\top + \Gamma V \Gamma^\top)$$

is controllable.

*Proof of Theorem 2*

We can make the variable substitution  $K = X\Sigma$  and maximize with respect to  $V$ ,  $K$  and  $\Sigma \succ 0$ . This gives the optimization problem

$$\begin{aligned}
&\sup_{V>0} \sup_{K, \Sigma \succ 0} Y \\
&\text{s. t. } P \geq K\Sigma^{-1}K^\top + V \\
&\quad \Sigma = F\Sigma F^\top + GG^\top - \Gamma Y \Gamma^\top \\
&\quad \Gamma = (FK^\top + F\Sigma H^\top + G)Y^{-1} \\
&\quad Y = K\Sigma^{-1}K^\top + KH^\top + HK^\top + H\Sigma H^\top \\
&\quad \quad + V + 1
\end{aligned}$$

Note that

$$P \geq K\Sigma^{-1}K^\top + V \geq K\Sigma^{-1}K^\top$$

Furthermore, the inequality  $P \geq K\Sigma^{-1}K^\top + V$  implies that

$$\begin{aligned}
Y &= K\Sigma^{-1}K^\top + KH^\top + HK^\top + H\Sigma H^\top + V + 1 \\
&\leq KH^\top + HK^\top + H\Sigma H^\top + P + 1
\end{aligned}$$

Thus, the optimization problem above is equivalent to

$$\begin{aligned}
&\sup_{K, \Sigma \succ 0} Y \\
&\text{s. t. } P \geq K\Sigma^{-1}K^\top \\
&\quad \Pi = FK^\top + F\Sigma H^\top + G \\
&\quad \Lambda = KH^\top + HK^\top + H\Sigma H^\top + P + 1 \\
&\quad \Sigma \preceq F\Sigma F^\top + GG^\top - \Pi\Lambda^{-1}\Pi^\top \\
&\quad Y \leq \Lambda
\end{aligned}$$

and we have eliminated the dependence on  $V$  of the optimisation problem. What has been gained with keeping  $V$  thus far is that we obtained an optimization problem over strictly positive definite matrices  $\Sigma$ .

The power inequality

$$P \geq K\Sigma^{-1}K^\top$$

is equivalent to the linear matrix inequality (LMI)

$$\begin{pmatrix} P & K \\ K^\top & \Sigma \end{pmatrix} \succeq 0$$

The error covariance inequality

$$\Sigma \preceq F\Sigma F^\top + GG^\top - \Pi\Lambda^{-1}\Pi^\top$$

can be recast as the LMI

$$0 \preceq \begin{pmatrix} F\Sigma F^\top - \Sigma + GG^\top & FK^\top + F\Sigma H^\top + G \\ (FK^\top + F\Sigma H^\top + G)^\top & \Lambda \end{pmatrix}$$

Summing up, the maximum entropy of the channel output is given by the value of the following maxmin problem

$$\begin{aligned}
&\sup_{K, \Sigma \succ 0} Y \\
&\text{s. t. } \\
&\quad 0 \preceq \begin{pmatrix} P & K \\ K^\top & \Sigma \end{pmatrix} \\
&\quad \Lambda = KH^\top + HK^\top + H\Sigma H^\top + P + 1 \\
&\quad 0 \preceq \begin{pmatrix} F\Sigma F^\top - \Sigma + GG^\top & FK^\top + F\Sigma H^\top + G \\ (FK^\top + F\Sigma H^\top + G)^\top & \Lambda \end{pmatrix} \\
&\quad Y \leq \Lambda
\end{aligned}$$

Clearly, we will have  $Y = \Lambda$  at optimality and we obtain the optimization problem

$$\begin{aligned}
&\sup_{K, \Sigma \succ 0} Y \\
&\text{s. t. } \\
&\quad 0 \preceq \begin{pmatrix} P & K \\ K^\top & \Sigma \end{pmatrix} \\
&\quad Y = KH^\top + HK^\top + H\Sigma H^\top + P + 1 \\
&\quad 0 \preceq \begin{pmatrix} F\Sigma F^\top - \Sigma + GG^\top & FK^\top + F\Sigma H^\top + G \\ (FK^\top + F\Sigma H^\top + G)^\top & Y \end{pmatrix} \tag{22}
\end{aligned}$$

*Matlab Code*

```

alpha = 0.7;
beta = -0.25;
sigma = sign(beta-alpha);
P = 1;
d = 1;
F = -beta;
G = 1;
H = alpha-beta;

a4 = alpha^2+beta^2*P;
a3 = 2*sigma*(alpha+beta*P);
a2 = P+1-alpha^2;
a1 = -2*sigma*alpha;
a0 = -1;
r = roots([a4 a3 a2 a1 a0]);

cvx_begin sdp
    variable S(d,d) symmetric
    variable K(1,d)
    variable Y
    S > 0
    [P K; K' S] > 0
    [F*S*F' - S + G*G' F*K' + F*S*H' + G;
     K*F' + H*S*F' + G' Y] > 0
    Y == K*H' + H*K' + H*S*H' + P + 1
    maximize Y;
cvx_end

C_sdp = 0.5*log2(Y)
C_poly = -log2(r(4))

```